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((F, D1), D3) bound state, S-duality and non-commutative open String/Yang-Mills theory

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Abstract: We study decoupling limits and S-dualities for non-commutative open string/Yang-Mills theory in a gravity setup by considering an $SL(2,\mathbb{Z})$ invariant supergravity solutions of the form ((F, D1), D3) bound state of type-IIB string theory. This configuration can be regarded as D3-brane solution with both electric and magnetic fields turned on along one of the spatial directions of the brane and preserves half of the space-time supersymmetries of the string theory. Our study indicates that there exists a decoupling limit for which the resulting theory is an open string theory defined in a geometry with noncommutativity in both space-time and space-space directions. We study S-duality of this non-commutative open string (NCOS) and find that the same decoupling limit in the S-dual description gives rise to a space-space non-commutative Yang-Mills theory (NCYM). We also discuss independently the decoupling limit for NCYM in this D3-brane background. Here we find that S-duality of NCYM theory does not always give a NCOS theory. Instead, it can give an ordinary Yang-Mills with singular metric and infinitely large coupling. We also find that the open string coupling relation between the two S-duality related theories is modified such that S-duality of a strongly coupled open-string/Yang-Mills theory does not necessarily give a weakly coupled theory. The relevant gravity dual descriptions of NCOS/NCYM are also given.

Keywords: D-branes, String Duality, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

Non-commutative geometry has appeared on various occasions in string theory starting from Witten's original proposal as a framework for open string field theory [1]. While this appearance might seem natural in the matrix theory formulation of string/M-theory [2]–[5], a more direct approach to extract the non-commutative nature of the underlying space in the conventional open string theory set up has been given in [6]–[10]. In these cases, the open string boundary condition in the presence of non-zero constant NSNS B-field plays a crucial role. In general, it is difficult to define a consistent quantum field theory in a non-commutative space-time because of the loss of manifest Lorentz covariance and causality. However, the fact that they can be embedded in string theory in a particular background suggests that they might exist as consistent quantum theory at least in some special cases. So, we ought to understand the nature of these theories in order to understand quantum gravity at very high energies where our notion of space-time changes drastically.

The appearance of non-commutative geometry particularly in space-space directions has been elucidated by Seiberg and Witten [10] in general setup by identifying a decoupling limit in which closed strings decouple from the open strings ending on Dp-branes in the presence of non-zero B-field (with only spatial components). The resulting theory is a Yang-Mills theory defined in non-commutative spaces. This decoupling limit has been studied for a special system by Hashimoto and Itzhaki [11] and also by Maldacena and Russo [12]. They studied D3-brane supergravity solution of type-IIB string theory with magnetic fields along one of the spatial directions of the brane. The supergravity solution in this case is nothing but the (D1, D3) bound state solution of type-IIB string theory, found in [13, 14], where there are infinite number of D-strings along, say x^1 direction of D3-brane (lying along, say x^1, x^2, x^3 directions). Under the decoupling limit, this (D1, D3) system provides the gravity dual description of $\mathcal{N} = 4$ SUSY non-commutative Yang-Mills theory (NCYM). The decoupling limit in this case corresponds to a purely field theoretic limit where the spatial directions x^2, x^3 become non commutative. The S-duality of the gravity dual description of NCYM has been considered by two of the present authors sometime ago[1] in [15], which, as we know now, provides the gravity dual description to the newly discovered non-commutative open string theory (NCOS) [16, 17].

On the other hand, if one considers D3-brane with an electric field along one of the spatial directions of the brane, the corresponding supergravity solution is given by (F, D3) bound state [13, 18] and is S-dual to (D1, D3) system. In this case there are infinite number of fundamental strings along, say, x^1 direction of D3-brane. It has been shown recently [16, 17] that even though it is not possible to obtain a field theory decoupling limit, a careful examination in this case reveals a decoupling limit in which the resulting theory is a non-commutative open string theory (NCOS) [16, 17]. This is consistent since in this case the coordinates x^0, x^1 become non commutative and a field theory with a non-commutative time coordinate cannot be unitary in general [19, 20, 21]. Moreover, it is shown in [17] that the S-dual of NCYM in the case with purely magnetic field gives just NCOS theory. The S-dual descriptions of NCOS in six and fewer spacetime dimensions are recently proposed and analysed in [22, 23, 24]. The gravity dual descriptions of these NCOS with pure electric fields are given in [25]. S-duality in non-commutative gauge theories has been discussed in [26, 27].

It is then natural to ask whether there exists a decoupling limit, when we consider D3-branes with both electric and magnetic fields turned on along the brane world-volume, such that the resulting theory decouples from the closed strings (or bulk supergravity). Also if such a decoupling limit exists, what kind of a theory does it correspond to and what is its strong coupling dual? We would like to address this issue based on a gravity consideration in this paper. In order to study the decoupling limit we consider the SL$(2, \mathbb{Z})$ invariant ((F, D1), D3) bound state solution of type-IIB string theory in two versions related by S-duality.[2] It is not a priori clear whether decoupling limits for this solution exist and whether the corresponding decoupled

[1] Unfortunately, we failed to recognize its conncetion to the non-commutative open string theory (NCOS). This same gravity dual description in the context of NCOS has also been given recently in [17].

[2] This solution has been constructed in [28]. There are infinite number of (p, q)-strings of Schwarz type along one of the spatial directions of D3-brane. Thus we note that the electric field and the magnetic field are parallel to each other and the solution preserves half of the spacetime supersymmetries of string theory.

theory is an NCYM or NCOS. We will first show that for D3-branes with both electric and magnetic fields, there exists a decoupling limit such that the corresponding theory can be described by an open string with both space-space and space-time noncommutativity. Then we will show that in the S-dual description the same NCOS decoupling limit always gives rise to NCYM with only space-space noncommutativity. We therefore show, in the present case, that the S-dual of NCOS theory is a NCYM theory, which is the converse to what has been shown in [17] for purely magnetic field case. Next we will describe that there also exists an independent decoupling limit, using this D3-brane configuration, for NCYM with space-space noncommutativity. However, we will show that this same decoupling limit in the S-dual description of NCYM does not always give rise to an NCOS, unlike the purely magnetic field case. For example, in some cases we end up with an ordinary Yang-Mills theory with a singular metric and an infinitely large gauge coupling. This is because the magnetic field in the dual description (or the electric field in the original description) modifies the would-be critical elecric field to a non-critical one eventhough with the electric field in the original description we have had a decoupling limit for nicely behaved NCYM. Let us clarify this point further to see how the magnetic field in the dual description controls the S-dual behaviour of NCYM. The ratio of electric field to the magnetic field in this case is of the order $1/\alpha'^{1+\beta}$ in the decoupling limit as we will show later, where α' is the fundamental string constant and the index $\beta \geq 0$. For $0 \leq \beta < 1$, the S-dual of NCYM is an ill-defined ordinary YM theory with a singular metric and an infinitely large gauge coupling. For $\beta = 1$, we end up with NCOS as the S-dual of NCYM with noncommutativity in both space-space and space-time directions. For $\beta > 1$, the S-dual of NCYM is also a NCOS but with only space-time noncommutativity. In other words, for $\beta > 1$, the effect of magnetic field on this theory becomes less important. For $\beta > 1$, the parameters used to define the NCOS or its gravity dual are independent of the actual value of β. This indicates that different scaling limits corresponding to different β with $\beta > 1$ are actually equivalent, i.e. giving the same NCOS.[3] For $\beta = 1$, the magnetic field is just right to have an effect on the NCOS such that it causes the space-space noncommutativity and modifies the relation between the open string coupling and its S-dual. For $0 \leq \beta < 1$, the magnetic field is too strong such that we do not even end up with a well defined theory. In particular, for $\beta = 0$, the electric field does not reach its critical value. So β plays the role of an order parameter. The critical value is $\beta = 1$. So NCYM and NCOS are related to each other by S-duality when $\beta \geq 1$. For $\beta > 1$, the coupling constants are related inversely to each other, therefore a strongly coupled theory is related to a weakly coupled theory by S-duality. For $\beta = 1$, this relation is modified and a strongly coupled theory does not necessarily give rise to a weakly coupled theory by S-duality.

[3] We thank the referee for pointing this out to us.

The organization of this paper is as follows. In section 2, we present the general setup for the quantities relevant for the decoupling limit in a flat background. We can take these quantities as the asymptotic values of various fields in a gravity configuration. In section 3, we first show how to obtain the two versions related by S-duality for the gravity configuration of ((F, D1), D3) bound state [28]. We then present these two versions explicitly for this bound state. In section 4, we first discuss the NCOS decoupling limit with space-space and space-time noncommutativity based on one version of the gravity description of ((F, D1), D3) bound state. We find that the corresponding decoupling limit in the S-dual version always gives NCYM. Therefore, we have the NCYM as the S-dual of the NCOS. We then discuss the decoupling limit for NCYM and study its S-duality. We find that the S-duality of NCYM does not necessarily give NCOS. In section 5, we present both the gravity dual description of the NCYM and that of NCOS in the respective decoupling limits. The conclusions drawn here agree completely with what we obtain in section 4. In section 6 we clarify some confusions which might arise in sections 4 and 5 and discuss possible quantizations of the open string coupling.

While preparing this manuscript we became aware of a paper [29] in which some related aspectes were also discussed.

2. The general set-up

We are considering D3-brane to lie along x^1, x^2, x^3 spatial directions and both electric and magnetic fields are turned on along the x^1 direction. Since the field strengths F_{01} (F_{23}) corresponding to electric (magnetic) fields can be suitably traded for NSNS B-field components B_{01} (B_{23}), so D3-brane can be regarded as living in a constant $g_{\mu\nu}$ and $B_{\mu\nu}$ background, where $\mu, \nu = 0, 1, 2, 3$. The consideration here simplifies the discussion of the decoupling limits presented in the following section. The closed string metric we consider has the form

$$-g_{00} = g_{11} = g_1 \qquad \text{and} \qquad g_{22} = g_{33} = g_2 \tag{2.1}$$

and the NS-NS field has the form

$$B_{01} = -B_{10} = b_1 \qquad \text{and} \qquad B_{23} = -B_{32} = b_2\,, \tag{2.2}$$

where g_1, g_2, b_1, b_2 are some constant parameters.

Because of the presence of $B_{\mu\nu}$ field, the open string boundary condition is not of purely Neumann type, but rather is a Neumann-Dirichlet mixed boundary condition [30] given as:

$$g_{\mu\nu}\partial_\sigma x^\nu + 2\pi\alpha' B_{\mu\nu}\partial_\tau x^\nu|_{\text{boundary}} = 0\,, \tag{2.3}$$

where ∂_σ and ∂_τ are respectively the normal and tangential derivatives to the world-sheet boundary. We stress that the coordinates x^μ defined in the above equation are the scaled ones used later in the paper.

The effective Seiberg-Witten open string metric is given by

$$G_{\mu\nu} = g_{\mu\nu} - (2\pi\alpha')^2 \left(B\, g^{-1}\, B\right)_{\mu\nu}. \tag{2.4}$$

Substituting $g_{\mu\nu}$ and $B_{\mu\nu}$ from eqs. (2.1) and (2.2) we obtain

$$-G_{00} = G_{11} \equiv G_1 = g_1 - \frac{(2\pi\alpha' b_1)^2}{g_1} = g_1 t(1 - \tilde{E}^2), \tag{2.5}$$

$$G_{22} = G_{33} \equiv G_2 = g_2 + \frac{(2\pi\alpha' b_2)^2}{g_2} = g_2(1 + \tilde{B}^2), \tag{2.6}$$

where dimensionless electric field $\tilde{E} = E/E_{\rm cr} = 2\pi\alpha' b_1/g_1$ with $E = b_1$ and the critical value of the electric field $E_{\rm cr} = g_1/2\pi\alpha'$. Similarly we have defined dimensionless magnetic field $\tilde{B} = B/B_0 = 2\pi\alpha' b_2/g_2$ with $B = b_2$ and $B_0 = g_2/2\pi\alpha'$.

The Seiberg-Witten relation for the antisymmetric non-commutativity parameter is

$$\Theta^{\mu\nu} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B}\right)_A^{\mu\nu}, \tag{2.7}$$

where "A" denotes the antisymmetric part. We find from (2.1) and (2.2)

$$\Theta^{01} \equiv \Theta_1 = \frac{-(2\pi\alpha')^2 b_1}{(2\pi\alpha' b_1)^2 - g_1^2} = \frac{\tilde{E}}{E_{\rm cr}(1 - \tilde{E}^2)}, \tag{2.8}$$

$$\Theta^{23} \equiv \Theta_2 = \frac{(2\pi\alpha')^2 b_2}{(2\pi\alpha' b_2)^2 + g_2^2} = \frac{\tilde{B}}{B_0(1 + \tilde{B}^2)}. \tag{2.9}$$

Also the open string coupling constant is given by

$$G_s = g_s \left(\frac{\det G_{\mu\nu}}{\det(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})}\right)^{1/2}, \tag{2.10}$$

where g_s denotes the closed string coupling constant. From eqs. (2.1), (2.2), (2.5) and (2.6) we obtain the open string coupling of the form

$$\begin{aligned} G_s &= g_s \left[1 - \frac{(2\pi\alpha' b_1)^2}{g_1^2}\right]^{1/2} \left[1 + \frac{(2\pi\alpha' b_2)^2}{g_2^2}\right]^{1/2} \\ &= g_s (1 - \tilde{E}^2)^{1/2} (1 + \tilde{B}^2)^{1/2}. \end{aligned} \tag{2.11}$$

It should be pointed out here that the dimensionless electric field $\tilde{E}$ cannot exceed 1, otherwsie the open string coupling would become imaginary. There is no such bound for the dimensionless magnetic field $\tilde{B}$. When $|\tilde{E}| \to 1$, the corresponding electric field attains a critical value, i.e. $|E| \approx E_{\rm cr}$ and then for a finite value of $\tilde{B}$, g_s has to scale appropriately to make the open string coupling G_s finite. Similarly when $|\tilde{E}| < 1$ and $\tilde{B} \to \infty$, g_s has to scale accordingly to make G_s finite. But when both $|\tilde{E}| \to 1$, and $\tilde{B} \to \infty$ g_s can remain finite if the two factors $(1-\tilde{E}^2)^{1/2}$ and $(1+\tilde{B}^2)^{1/2}$ compensate each other to make open string coupling in (2.11) finite. We will point out what the resulting theory is in each of the cases in the following sections. Unlike in purely electric case, a critical electric field does not always give rise to a NCOS.

Finally we note that the effective open string metric (2.4) and the non-commutativity parameter (2.7) has been obtained by looking at the disk propagator at the boundary [31, 10]:

$$\langle x^\mu(\tau) x^\nu(0)\rangle = -\alpha' G^{\mu\nu} \ln \tau^2 + i\frac{\Theta^{\mu\nu}}{2}\epsilon(\tau)\,. \tag{2.12}$$

We mention here that, in the decoupling limit, when the second term in the r.h.s. of (2.12) remains finite while the first term goes to zero as $\alpha' \to 0$, we get an NCYM theory as in (D1, D3) (i.e. purely magnetic case). But when both the terms remain finite then we get NCOS theory as in (F, D3) case. We will see in the folowing sections how these two cases can be realised in the decoupling limit of the same ((F, D1), D3) bound state, i.e. D3-brane with both electric and magnetic fields.

3. ((F, D1), D3) bound state

As we mentioned earlier ((F, D1), D3) bound state solution of type-IIB string theory corresponds to D3-brane with both electric and magnetic field turned on along one of the spatial directions of the brane. This solution has been explicitly constructed in [28] and is known to preserve half of the spacetime supersymmetries of string theory. For the purpose of this paper, we need to write this gravity configuration in two versions related by S-duality. Since we have non-vanishing RR scalar χ, the S-duality cannot be implemented in a simple fashion as is usually done. The main reason is that the dilaton does not go to its inverse under S-duality in the presence of χ. Here we adopt the following approach. We first write down the manifestly $SL(2,\mathbb{Z})$ covariant ((F, D1), D3) configuration in the Einstein frame for the metric, dilaton, and the NSNS B-field in the NSNS sector and the RR scalar χ, RR 2-form A_2 and RR 4-form A_4 with its self-dual field strength F_5 in the RR sector as:

$$\begin{aligned}
dS_E^2 &= e^{U_0}(HH')^{1/4}\Big[H^{-1}\left(-(dx^0)^2+(dx^1)^2\right)+H'^{-1}\left((dx^2)^2+(dx^3)^2\right)+\\
&\qquad\qquad + dr^2 + r^2 d\Omega_5^2\Big]\,,\\
e^{\phi} &= g_s\frac{H''}{\sqrt{HH'}}\,,\\
\chi &= \frac{pq(H-H')+g_s\chi_0\,\Delta_{(p,q)}H'}{q^2H+g_s^2(p-\chi_0 q)^2H'}\,,
\end{aligned}$$

$$
\begin{aligned}
2\pi\alpha' B &= g_s e^{U_0}(p-\chi_0 q)\,\Delta_{(p,q,n)}^{-1/2} H^{-1} dx^0\wedge dx^1 - \frac{q}{n}H'^{-1}dx^2\wedge dx^3\,,\\
A_2 &= e^{U_0}\left[q g_s^{-1} - \chi_0(p-\chi_0 q)\right]\Delta_{(p,q,n)}^{-1/2} H^{-1} dx^0\wedge dx^1 + \frac{p}{n}dx^2\wedge dx^3\,,\\
F_5 &= 16\pi n\alpha'^2(\star\epsilon_5+\epsilon_5)\,, \qquad (3.1)
\end{aligned}
$$

where harmonic functions H, H' and H'' are defined as

$$
\begin{aligned}
H &= 1+\frac{Q_3}{r^4}\,, \qquad H' = 1+\frac{n^2 e^{2U_0}}{\Delta_{(p,q,n)}}\frac{Q_3}{r^4}\,,\\
H'' &= 1+\frac{g_s^{-1}q^2+e^{2U_0}n^2}{\Delta_{(p,q,n)}}\frac{Q_3}{r^4}\,. \qquad (3.2)
\end{aligned}
$$

In the above, p, q, n are integers.[4] n represents the number of D3-branes and is inert under S-duality (or in general under SL$(2,\mathbb{Z})$). There are actually an infinite number of (p,q) strings of Schwarz type in D3-brane world volume. To be precise, there is a single (p,q) string per $(2\pi)^2\alpha'$ area of the infinite x^2, x^3 plane of the D3-brane (note that (p,q) strings lie along x^1). (p,q) transforms as a doublet under SL$(2,\mathbb{Z})$. In particular, $p\to\hat{p}=q$ and $q\to\hat{q}=-p$ under S-duality. Also, $r=\sqrt{(x^4)^2+\cdots+(x^9)^2}$, $d\Omega_5^2$ is the line element on unit 5-sphere, $\star$ denotes the Hodge dual in 10-dimensions, ϵ_5 is the volume form defined on the 5-sphere of unit radius, and g_s is the closed string coupling constant and is given as $g_s=e^{\phi_0}$ with ϕ_0 the asymptotic value of the dilaton. χ_0 is the asymptotic value for the RR scalar and we set it to zero for the rest of this paper. This implies that for the asymptotic closed string coupling, we still have $g_s\to 1/g_s$ under S-duality (but not for the general coupling e^ϕ). The constant U_0 is inert under SL$(2,\mathbb{Z})$. This constant is chosen for the purpose of labeling different vacua of non-perturbative type-IIB string theory.[5] For example, $U_0=0$ corresponds to the supergravity vaccum while $U_0=-\phi_0/2$ corresponds to the vacuum for the perturbative type-IIB string theory. Usually, when we choose a special U_0 (or vacuum), we break the SL$(2,\mathbb{Z})$ symmetry manifestly such as the above perturbative type-IIB string vaccum. With this U_0, the $Q_3, \Delta_{(p,q)}$ and $\Delta_{(p,q,n)}$ are all SL$(2,\mathbb{Z})$ invariant and are given as

$$
\begin{aligned}
\Delta_{(p,q)} &= g_s(p-\chi_0 q)^2 + g_s^{-1}q^2\,, \qquad \Delta_{(p,q,n)} = \Delta_{(p,q)} + e^{2U_0}n^2\,,\\
Q_3 &= 4\pi\alpha'^2\Delta_{(p,q,n)}^{1/2}e^{-3U_0}\,. \qquad (3.4)
\end{aligned}
$$

Note that the string constant α' is also inert under SL$(2,\mathbb{Z})$ which is consistent with the expression for the self-dual 5-form field strength given in (3.1). With the above,

[4] We can replace the integral charges p and q in the above equations by the trigonometric functions of two angles defined as:

$$
\cos\theta = \frac{e^{U_0}n}{\sqrt{g_s^{-1}q^2+e^{2U_0}n^2}}\,, \qquad \cos\alpha = \frac{\sqrt{g_s^{-1}q^2+e^{2U_0}n^2}}{\sqrt{g_s p^2+g_s^{-1}q^2+e^{2U_0}n^2}}\,. \qquad (3.3)
$$

We will make use of these definitions later.

[5] The asymptotic region of a gravity configuration corresponds to such a vaccum.

one can check easily that the harmonic functions H and H' are $\mathrm{SL}(2,\mathbb{Z})$ invariant but the harmonic function H'' is not. These are consistent with the Einstein-frame metric (which is $\mathrm{SL}(2,\mathbb{Z})$ invariant) and the expression for the dilaton. The transformation of the harmonic function H'' under $\mathrm{SL}(2,\mathbb{Z})$ is determined by that of the dilaton.

With the above, we can easily write down the gravity configuration for the ((F, D1), D3) bound state in two versions related by S-duality. They would look the same in form as given above in eq. (3.1). If we denote the above configuration as version A and denote the corresponding fields in the S-dual version (called version B) with a hat over the fields,[6] then the S-dual version (i.e. version B) is related to version A through the following relations (for vanishing χ_0):

$$\hat{p} = q\,, \qquad \hat{q} = -p\,, \qquad \hat{g}_s = \frac{1}{g_s}\,. \tag{3.5}$$

Note that the constant U_0, the harmonic functions H and H' and the 5-form field strength F_5 remain the same in the two versions while the rest of the fields are related as

$$\begin{aligned} e^{\hat{\phi}} &= g_s^{-1}\frac{\hat{H}''}{\sqrt{HH'}}\,, \qquad \hat{\chi} = -\frac{pq(H-H')}{p^2H + g_s^{-2}q^2H'}\,, \\ 2\pi\alpha'\hat{B} &= A_2\,, \qquad \hat{A}_2 = -2\pi\alpha' B\,, \end{aligned} \tag{3.6}$$

where the harmonic function $\hat{H}''$ is now given as

$$\hat{H}'' = 1 + \frac{g_s p^2 + e^{2U_0}n^2}{\Delta_{(p,q,n)}}\frac{Q_3}{r^4}\,. \tag{3.7}$$

Our purpose in this paper is to study decoupled theories and their respective S-dualities in type-IIB string theory. So we should use the string-frame rather than the Einstein-frame description of the above configuration. In version A, we have the string-frame metric as $ds^2 = e^{\phi/2}ds_E^2$ while in version B, we have $d\hat{s}^2 = e^{\hat{\phi}/2}ds_E^2$ with ds_E^2 the above Einstein-frame metric. Given the asymptotic region of the gravity configuration as a vacuum of the underlying string theory, we can choose, as is usually done, either of the two S-duality related string-frame metric to have the form η_{MN} with $\eta_{MN} = (-1, 1, \ldots, 1)$ $(M, N = 0, 1, \ldots, 9)$ asymptotically. To be specific, we impose this on version A. This choice breaks the $\mathrm{SL}(2,\mathbb{Z})$ symmetry manifestly as is evident from $U_0 = -\phi_0/2$. This can also be understood from the fact that we have made a preferable choice for strings over the other objects such as 3-branes and 5-branes in this theory.

[6] We use $\hat{A}$ to denote the field A in the S-dual version.

With the above, the gravity configuration in version A can be re-expressed as:

$$\begin{aligned} ds^2 &= H''^{1/2}\Big[H^{-1}\left(-(dx^0)^2+(dx^1)^2\right)+H'^{-1}\left((dx^2)^2+(dx^3)^2\right)+ \\ &\qquad + dr^2 + r^2 d\Omega_5^2\Big]\,, \\ e^{2\phi} &= g_s^2\frac{H''^2}{HH'} \qquad \chi = \frac{g_s^{-1}\sin\theta\tan\alpha\,(H-H')}{H\sin^2\theta + H'\tan^2\alpha}\,, \\ 2\pi\alpha' B &= \sin\alpha\, H^{-1}dx^0\wedge dx^1 - \tan\theta H'^{-1}dx^2\wedge dx^3\,, \\ A_2 &= g_s^{-1}\sin\theta\cos\alpha\, H^{-1}dx^0\wedge dx^1 + g_s^{-1}\tan\alpha\cos^{-1}\theta H'^{-1}dx^2\wedge dx^3\,, \\ F_5 &= 16\pi n\alpha'^2(\star\epsilon_5+\epsilon_5)\,, \end{aligned} \tag{3.8}$$

where the harmonic functions are

$$\begin{aligned} H &= 1+\frac{4\pi g_s n\alpha'^2}{r^4}\frac{1}{\cos\theta\cos\alpha}\,, \\ H' &= 1+\frac{4\pi g_s n\alpha'^2}{r^4}\cos\theta\cos\alpha\,, \\ H'' &= 1+\frac{4\pi g_s n\alpha'^2}{r^4}\frac{\cos\alpha}{\cos\theta}\,, \end{aligned} \tag{3.9}$$

with

$$\cos\theta = \frac{n}{\sqrt{q^2+n^2}}\,, \qquad \cos\alpha = \frac{\sqrt{q^2+n^2}}{\sqrt{(pg_s)^2+q^2+n^2}}\,. \tag{3.10}$$

The S-dual description (i.e. version B) of the above can be written as:

$$\begin{aligned} d\hat{s}^2 &= \hat{g}_s\hat{H}''^{1/2}\Big[H^{-1}\left(-(dx^0)^2+(dx^1)^2\right)+H'^{-1}\left((dx^2)^2+(dx^3)^2\right)+dr^2+r^2d\Omega_5^2\Big]\,, \\ e^{2\hat{\phi}} &= \hat{g}_s^2\frac{\hat{H}''^2}{HH'}\,, \\ \hat{\chi} &= -\hat{g}_s^{-2}\frac{H''}{\hat{H}''}\chi\,, \end{aligned} \tag{3.11}$$

where harmonic functions $H(=\hat{H}), H'(=\hat{H}')$ and H'' are the same as in version A,[7] $\hat{g}_s = 1/g_s$ with g_s the closed string coupling in the original version and the harmonic function $\hat{H}''$ is not SL$(2,\mathbb{Z})$ invariant (neither is H'') and is given as

$$\hat{H}'' = 1+\frac{4\pi n g_s\alpha'^2}{r^4}\cos\alpha\cos\theta\left(1+\frac{\tan^2\alpha}{\cos^2\theta}\right)\,. \tag{3.12}$$

We also have $2\pi\alpha'\hat{B} = A_2$ and $\hat{A}_2 = -2\pi\alpha' B$ with A_2 and B given in (3.8). The five-form remains the same as in the original version. For later comparison, we give

[7] Because of our special choice $U_0 = -\phi_0/2$, harmonic functions H and H' do not appear to be manifestly invariant under S-duality.

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the angles in the dual frame as,

$$\cos\hat\theta = \frac{n}{\sqrt{g_s^2 p^2 + n^2}} = \left(1 + \frac{\tan^2\alpha}{\cos^2\theta}\right)^{-1/2},$$
$$\cos\hat\alpha = \frac{\sqrt{g_s^2 p^2 + n^2}}{\sqrt{g_s^2 p^2 + q^2 + n^2}} = \left(1 + \frac{\tan^2\alpha}{\cos^2\theta}\right)^{1/2} \cos\theta\cos\alpha\,. \tag{3.13}$$

4. The decoupling limit and S-duality

In this section we will be looking for decoupling limits such that the D3-brane in the presence of both electric and magnetic fields decouples from the bulk closed strings (or bulk gravity) and at the same time it gives rise to sensible quantum theories for the decoupled D3-brane. This requires the open string coupling to remain fixed in the decoupling limit. In the following, we first discuss NCOS decoupling limit and study its S-duality (We will refer this as case (I)). Then we will discuss NCYM decoupling limit and its S-duality (We call this as case (II)).

4.1 Case I: NCOS decoupling limit and S-duality

Since we have given two versions of the gravity configuration for the ((F, D1), D3) bound state related by S-duality in the previous section, we should be able to show if there exists a connection between NCYM and NCOS by using purely the gravity description (if such decoupled theories exist at all). In the case of purely magnetic field on D3-brane it has been shown by Gopakumar et al. [17] that starting from the known NCYM limit, one can use the S-duality and gauge transformation on the background B-field to get an NCOS theory. Thus in this case the S-dual of NCYM is a NCOS theory. In the following, we will show that in the presence of both electric and magnetic fields, the decoupling limit for NCOS in version A corresponds to a decoupling limit for NCYM in version B. This implies that the S-dual of NCOS is NCYM even in the present case. However, the converse is not quite true, i.e. the S-duality of NCYM does not always give an NCOS, as we will demonstrate later in subsection 4.2.

Let us start with the ((F, D1), D3) configuration in version A (eq. (3.8)). The fields discussed in section 2 correspond to the asymptotic values of the respective fields in this gravity configuration. For general purpose, we assume

$$x^{0,1} = \sqrt{g_1}\tilde{x}^{0,1}\,, \qquad x^{2,3} = \sqrt{g_2}\tilde{x}^{2,3}\,, \tag{4.1}$$

where $\tilde{x}^\mu$ for $\mu = 0, 1, 2, 3$ remain fixed in the decoupling limit. We then have

$$\tilde{E} = \frac{2\pi\alpha' b_1}{g_1} = \sin\alpha\,, \qquad \tilde{B} = \frac{2\pi\alpha' b_2}{g_2} = -\tan\theta\,, \tag{4.2}$$

where $b_1 = g_1 \sin\alpha/(2\pi\alpha')$ and $b_2 = -g_2 \tan\theta/(2\pi\alpha')$. Using (2.5), (2.6), (2.8), (2.9) and (2.10), we have

$$\begin{aligned}
-G_{00} &= G_{11} = G_1 = g_1 \cos^2\alpha\,, \qquad G_{22} = G_{33} = G_2 = \frac{g_2}{\cos^2\theta}\,,\\
\Theta^{01} &= \Theta_1 = \frac{2\pi\alpha' \sin\alpha}{g_1 \cos^2\alpha}\,, \qquad \Theta^{23} = \Theta_2 = -\frac{2\pi\alpha' \sin\theta\cos\theta}{g_2}\,,\\
G_s &= g_s \frac{\cos\alpha}{\cos\theta}\,.
\end{aligned} \tag{4.3}$$

In order to have NCOS limit, we need at least the non-commutative parameter Θ_1 in (4.3) to be non vanishing in the decoupling limit $\alpha' \to 0$, which requires $g_1 \cos^2\alpha \sim \alpha'$. So $\alpha' G^{00} = -\alpha' G^{11}$ remain fixed and therefore the resulting theory will be stringy rather than a field theory according to eq. (2.12) if the corresponding decoupling limit exists. Since $\cos^2\alpha \le 1$, we must have $g_1 \sim \alpha'^{\delta}$ with $\delta \le 1$. We limit ourselves to $\delta < 1$ since we do not have decoupling for the special case $\delta = 1$.[8] Let us now examine the metric in version A. The decoupling limit requires that the near-horizon region decouples from the asymptotic flat region. In other words, we need the metric describing the near-horizon region to scale homogeneously in certain power of α'. This uniquely determines the scalings in terms of α' for all the relevant quantities except for the A_{01} component of the RR 2-form A_2 if $\delta < 1$. For the present purpose, we list the decoupling limit collectively in the following:

$$\begin{aligned}
r &= \sqrt{\tilde{b}'\alpha'}\,u\,, \qquad g_2 = \frac{\alpha'}{\tilde{b}'}\,, \qquad g_1 = \left(\frac{\alpha'}{\tilde{b}'}\right)^{\delta},\\
\cos\theta &= \frac{\tilde{b}}{\tilde{b}'}\,, \qquad \cos^2\alpha = \left(\frac{\alpha'}{\tilde{b}'}\right)^{1-\delta},
\end{aligned} \tag{4.4}$$

where u, $\tilde{b}$ and $\tilde{b}'$ all remain fixed and $\delta < 1$. The above condition $\cos\theta = \tilde{b}/\tilde{b}'$ is needed only for NCOS with noncommutativity in both space-space and space-time directions. To include all the cases of NCOS, we should use $\sin\theta = c(\alpha'/\tilde{b})^{\delta'}$ instead, with constant $c \le 1$ and parameter $\delta' \ge 0$. As will be shown in section 6, the $\delta' \ge 0$ corresponds to $\beta \ge 1$ which will be discussed in the following subsection. In particular, $\delta' = 0$ corresponds to $\beta = 1$. The $\delta' = 0$ is the same condition as $\cos\theta = \tilde{b}/\tilde{b}'$. But for $\delta' > 0$, we have the non-commutative parameter Θ_2 vanishing and so we end up with noncommutativity only in space-time directions, corresponding to vanishing magnetic field. Here we just concentrate on $\delta' = 0$ case.

[8]For $\delta = 1$, $\cos\alpha$ remains fixed as $\alpha' \to 0$. Given fixed open string coupling $G_s = g_s \cos\alpha/\cos\theta$, the string metric can only scale homogeneously in α' if $r \sim \sqrt{\alpha'}$, $g_2 \sim \alpha'$ and $\cos\theta$ remains fixed. Then g_s also remains fixed. Since $\alpha' G^{\mu\nu}$ remain fixed for $\mu, \nu = 0, 1, 2, 3$ and $\tilde{E} = \sin\alpha$ is fixed and not equal to unity, we do not expect to have a decoupled open string theory. What is interesting in this case is that both non-commutative parameters Θ_1 and Θ_2 are non-vanishing even as we take $\alpha' \to 0$. In other words, in the near horizon region, the open string feels the geometry of the D3-brane to be non commutative.

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If we calculate $G_1, G_2, \Theta_1, \Theta_2$ and $\tilde{E}$, using eqs. (4.2), (4.3) and (4.4), we have

$$\begin{aligned} G_1 &= \frac{\alpha'}{\tilde{b}'}\,, \qquad G_2 = \frac{\alpha'\tilde{b}'}{\tilde{b}^2}\,, \\ \Theta_1 &= 2\pi\tilde{b}'\,, \qquad \Theta_2 = -2\pi\frac{\tilde{b}}{\tilde{b}'}\sqrt{\tilde{b}'^2 - \tilde{b}^2}\,, \\ \tilde{E} &= \sin\alpha = 1 - \frac{1}{2}\left(\frac{\alpha'}{\tilde{b}'}\right)^{1-\delta}\,. \end{aligned} \tag{4.5}$$

Since Θ_1 and Θ_2 remain fixed (note that Θ_2 vanishes in the special case of $\tilde{b} = \tilde{b}'$, corresponding to vanishing magnetic field), we have noncommutativity in both space-space and space-time directions. Further we have $\tilde{E} \to 1$ as $\alpha' \to 0$ (since $\delta < 1$), i.e. electric field reaching its critical value. We therefore expect a decoupled theory. Since both $\alpha' G_1^{-1}$ and $\alpha' G_2^{-1}$ remain fixed, by looking at the correlator in eq. (2.12), we must conclude that this decoupled theory is an open string theory (rather than a field theory) defined on a spacetime with space-space and space-time non-commutative geometry. Therefore, in the decoupling limit (4.4), the ((F, D1), D3) system is described by a space-space and space-time non-commutative open string theory. Open strings are lying along x^1-direction, and x^0, x^1 coordinates are non-commuting (as Θ_1 =finite) as in purely electric field case discussed by Seiberg et al. [16] and Gopakumar et al. [17]. However, unlike in that case, we also have x^2, x^3 coordinates non commuting (as Θ_2 =finite). As in purely electric case here also open strings cannot bend to form closed strings, i.e. closed strings completely decouple. This can be understood as discussed by Seiberg et al. [16], from the expression of the electric field at critical value $E \approx E_{\rm cr} \sim \alpha'^{\delta-1} \to \infty$. So, it would require an infinite amount of energy to bend such a string and therefore closed strings cannot be formed. Another way to understand the decoupling of the open string from the bulk closed strings is to compare the relevant energy scales. The closed string scale is $M_s = 1/\sqrt{\alpha'}$ while the NCOS scale is determined by the non-commutative parameter Θ_1 as $M_{\rm eff} \sim 1/\sqrt{\tilde{b}'}$. This NCOS scale can be determined from: $1/(4\pi\alpha')\int \partial\tilde{x}^1\partial\tilde{x}^1 G_{11} = 1/(4\pi\tilde{b}')\int \partial\tilde{x}^1\partial\tilde{x}^1$. In other words, the effective $\alpha'_{\rm eff} = \tilde{b}'$. In the limit $\alpha' \to 0$, the former becomes infinite while the latter is fixed, therefore the open string decouples from the closed strings. We can further understand this in the following way. Using $\alpha'_{\rm eff}$, the metric for NCOS can be scaled to $G_{\mu\nu} = \eta_{\mu\nu}$. The mass spectrum for NCOS is

$$\alpha'_{\rm eff} M = \alpha'_{\rm eff}\left(p_0^2 - p_1^2 - p_2^2 - p_3^2\right) = N - 1\,, \tag{4.6}$$

where for simplicity we consider only bosonic string and N is the number of string excitations. Given that $\alpha'_{\rm eff}$ is fixed as $\alpha' \to 0$, the above equation is consistent with the non-zero $\alpha' G^{-1}$ in the two-point function (2.12). In other words, we have undecoupled massive open string states from the massless ones and the energies of these finite exciations remain fixed in the decoupling limit. Let us see if any of these open strings can be away from the brane and turn into closed strings. If this is true, we should have the mass spectrum for the closed strings (using the closed string metric) as

$$\tilde{b}' \left(\frac{\alpha'}{\tilde{b}'}\right)^{1-\delta} (p_0^2 - p_1^2) - \tilde{b}' \left(p_2^2 + p_3^2\right) = 2N + 2\bar{N} - 4\,. \tag{4.7}$$

The above equation cannot be satisfied unless the energy goes to infinity as $\alpha' \to 0$ (since $\alpha'^{1-\delta} \to 0$). We therefore conclude that any NCOS with finite energy decouples from the closed strings and is confined to the branes.

We like to point out that the NCOS discussed above is insensitive to the parameter δ even though the scaling limits are. This can be understood by the fact that the NCOS is defined by the effective open string metric, the non-commutative parameters and the open string coupling, none of which has any dependence on the δ parameter. This is also true for the gravity dual description of NCOS which will be discussed in the following section. Therefore, the scaling limits for different $\delta < 1$ appears equivalent.

Let us now examine what this NCOS would look like in the S-dual theory, i.e. in version B. From the dual string metric and dual B-field, we have the following:

$$\begin{aligned}
-\hat{G}_{00} &= \hat{G}_{11} = \hat{G}_1 = g_1 g_s^{-1}\left(1 - \sin^2\theta\cos^2\alpha\right),\\
\hat{G}_{22} &= \hat{G}_{33} = \hat{G}_2 = g_2 g_s^{-1}\left(1 + \frac{\tan^2\alpha}{\cos^2\theta}\right),\\
\hat{\Theta}^{01} &= \hat{\Theta}_1 = \frac{2\pi\alpha'\sin\theta\cos\alpha}{g_1 g_s^{-1}\left(1 - \sin^2\theta\cos^2\alpha\right)}, \qquad \hat{\Theta}^{23} = \hat{\Theta}_2 = \frac{2\pi\alpha'\tan\alpha/\cos\theta}{g_2 g_s^{-1}\left(1 + \frac{\tan^2\alpha}{\cos^2\theta}\right)},\\
\hat{G}_s &= g_s^{-1}\left(1 - \sin^2\theta\cos^2\alpha\right)^{1/2}\left(1 + \frac{\tan^2\alpha}{\cos^2\theta}\right)^{1/2},\\
\tilde{\hat{E}} &= \sin\theta\cos\alpha\,, \qquad \tilde{\hat{B}} = \frac{\tan\alpha}{\cos\theta}\,.
\end{aligned} \tag{4.8}$$

Using the decoupling limit in (4.4), we have

$$\begin{aligned}
&\hat{G}_1 = G_s^{-1}\frac{\tilde{b}'}{\tilde{b}}\left(\frac{\alpha'}{\tilde{b}'}\right)^{(1+\delta)/2}, \qquad \hat{G}_2 = G_s^{-1}\left(\frac{\tilde{b}'}{\tilde{b}}\right)^3\left(\frac{\alpha'}{\tilde{b}'}\right)^{(1+\delta)/2},\\
&\hat{\Theta}_1 = 0\,, \qquad \hat{\Theta}_2 = 2\pi G_s\frac{\tilde{b}^2}{\tilde{b}'}\,, \qquad \hat{G}_s = \frac{1}{G_s}\left(\frac{\tilde{b}'}{\tilde{b}}\right)^2,\\
&\tilde{\hat{E}} \sim \alpha'^{(1-\delta)/2}\,, \qquad \tilde{\hat{B}} \sim \alpha'^{-(1-\delta)/2}\,.
\end{aligned} \tag{4.9}$$

As $\alpha' \to 0$, $\tilde{\hat{E}} \to 0$ and $\tilde{\hat{B}} \to \infty$ ($\hat{b}_2$ fixed). Further we have $\alpha' \hat{G}_1^{-1} \sim \alpha' \hat{G}_2^{-1} \sim \alpha'^{(1-\delta)/2} \to 0$, since $\delta < 1$, and the open string coupling $\hat{G}_s$ and non-commutative parameter $\hat{\Theta}_2$ are all fixed. Therefore, we end up with NCYM as expected. Notice that for $\delta = -1$, $\hat{G}_1$ and $\hat{G}_2$ are also fixed.[9] The open string coupling relation given in (4.9) gives $\hat{G}_s G_s = (\tilde{b}'/\tilde{b})^2 > 1$, since $\tilde{b}'/\tilde{b} = 1/\cos\theta > 1$. This implies that a strongly coupled theory does not necessarily give rise to a weakly coupled theory after S-duality unless the couplings for the strongly coupled theory is greater than $(\tilde{b}'/\tilde{b})^2$ in the presence of both electric and magnetic fields. This is quite different from the purely electric or magnetic field case. The reason for this is actually simple. In the presence of both electric and magnetic fields, we have the term $F \wedge F$ non vanishing. In other words, we can effectively have an axion coupling. As we know, with non-vanishing axion, the coupling in the S-dual theory is not inversely related to the coupling in the original theory. We will demonstrate this in the gravity dual description by using the fact that the open string coupling is the same as the closed string coupling at the IR and the fact that there is an induced S-duality in the field theory from the S-duality of type-IIB string theory. In summary, we have shown that the S-duality of NCOS in the presence of both electric and magnetic fields gives an NCYM theory, like in the purely electric case [17]. Using the effective description discussed in the previous footnote, NCYM is essentially inert to the parameter $\delta < 1$. The decoupling of the NCYM from the bulk gravity is the usual one and we do not repeat the discussion here. In the following, we want to know whether the converse is true.

[9]For other values of $\delta < 1$, the metric for NCYM appears to be singular while the coupling constant remains fixed. Note that one cannot simply rescale the coordinates $\tilde{x}^\mu$ to make the metric nonsingular anymore because we assume the coordinates $\tilde{x}^\mu$ to be fixed from the outset. Further, the two-point function (2.12) is defined with respect to these fixed coordinates. If we rescale $\tilde{x}^\mu$ to $\bar{x}^\mu$ in order to have a nonsingular metric, we would end up with $< \bar{x}^\mu \bar{x}^\nu > \sim 0$, an ordinary field theory rather than a non-commutative one. From (4.9), we see that the singular factor in the metric is $(\alpha'/\tilde{b}')^{(1+\delta)/2}$ which is also the singular factor appearing in the asymptotic (i.e. $r \to \infty$) closed string metric in version B. In other words, in the decoupling limit, unlike the usual case corresponding to $\delta = -1$, the asymptotic closed string metric is not flat Minkowski rather a flat Minkowski times the above singular factor $(\alpha'/\tilde{b}')^{(1+\delta)/2}$. So, the singular behavior of the open string metric is being inherited from that of the asymptotic closed string metric. As we know, the closed string is quantized perturbatively with respect to the flat Minkowski vacuum, which usually corresponds to the asymptotic region of a gravity configuration. In the present case, we could either choose $\delta = -1$ with string constant α' or we can have an effective description in the sense that we have an effective string constant $\alpha'_{\text{eff}} = \tilde{b}'(\alpha'/\tilde{b}')^{(1-\delta)/2}$ (which can be obtained from $1/(4\pi\alpha') \int \partial x^M \partial x^N g_{MN}(r \to \infty) = 1/[4\pi\tilde{b}'(\alpha'/\tilde{b}')^{(1-\delta)/2}] \int \partial y^M \partial y^N \eta_{MN}$, with $y^{0,1} = x^{0,1}/\sqrt{g_1} = \tilde{x}^{0,1}$, $y^{2,3} = x^{2,3}/\sqrt{g_1}$, $y^{4,\dots,9} = x^{4,\dots,9}/\sqrt{g_1}$), with again flat Minkowski metric $\eta_{MN} = (-,+,\dots,+)$. Using this effective description, we calculate again the two-point function (2.12) (with respect to the same fixed $\tilde{x}^\mu$) and find that the new open string metric $(\hat{G}_{\text{eff}})_{\mu\nu}$ is nonsingular and is related to the original string metric as $\alpha' \hat{G}^{\mu\nu} = \alpha'_{\text{eff}} \hat{G}^{\mu\nu}_{\text{eff}}$. Using the effective open string metric $(\hat{G}_{\text{eff}})_{\mu\nu}$, the NCYM is well-defined and is independent of the parameter $\delta < 1$. This is true also for the gravity dual description which will be discussed in the following section. In other words, many of these scaling limits corresponding to different $\delta < 1$ are actually equivalent.

4.2 Case II: NCYM decoupling limit and S-duality

In this case we discuss the decoupling limit for NCYM for the configuration ((F, D1), D3) in version A (eq. (3.8)). To be specific, we insist[10] that $G_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) remain fixed (we choose them to be normalized to $\eta_{\mu\nu}$) as $\alpha' \to 0$. In order to have NCYM, we need Θ_2 and $b_2 = -g_2 \tan\theta/(2\pi\alpha')$ to remain fixed. From (4.3), we have the following scalings:

$$g_1 \cos^2\alpha = 1\,, \qquad g_2 = \left(\frac{\alpha'}{\tilde{b}}\right)^2, \qquad \cos\theta = \frac{\alpha'}{\tilde{b}}\,, \tag{4.10}$$

where $\tilde{b}$ is a fixed parameter and is not related to the $\tilde{b}$ in the previous section. We also insist that the open string coupling G_s to be fixed which can be used to determine the scaling behavior of the closed string coupling g_s. With $g_1 \cos^2\alpha = 1$, one can check from (4.3) that the non-commutative parameter Θ_1 always vanishes as $\alpha' \to 0$. In the decoupling limit, the gravity dual description of NCYM in the near horizon region decouples from the asymptotic region. This requires that the near-horizon metric scales homogeneously in terms of certain power of α'. It is not difficult to check that for any g_1 satisfying $g_1 \cos^2\alpha = 1$, the scaling behavior for the radial coordinate can be determined uniquely as

$$r = \alpha' u\,, \tag{4.11}$$

with u fixed. We would like to point out one special case for which the closed string coupling g_s remains unscaled. From $G_s = g_s \cos\alpha/\cos\theta$, the choice $\cos\alpha \sim \alpha'$ will do the job.

Let us now examine the S-duality of the NCYM. For $g_1 \cos^2\alpha = 1$, we have three cases which can be studied:

(1) g_1 and $\cos\alpha (\neq 1)$[11] are both fixed and independent of the limit: $\alpha' \to 0$,

(2) $g_1 = (\alpha'/\tilde{b}')^\delta$ with $\delta < 0$ and $\cos\alpha = (\alpha'/\tilde{b}')^{-\delta/2}$

(3) $g_1 \to 1$ and $\sin\alpha = (\alpha'/\tilde{b}')^\beta$ with $\beta > 0$.

Note here that the parameters δ and $\tilde{b}'$ are different than those discussed in the previous case. From (4.8), we know that the quantity determining whether we have NCOS in the dual theory or not is $\tilde{\hat{E}} = \sin\theta\cos\alpha$. Only for $\tilde{\hat{E}} \to 1$ as $\alpha' \to 0$, we can potentially have decoupled NCOS. From the above decoupling limit for NCYM, we

[10] We could give a more general discussion by insisting only $\alpha' G^{\mu\nu} \to 0$, i.e. for a field theory.

[11] The $\cos\alpha = 1$ corresponds to vanishing electric field which is not our interest here. We want to have both electric and magnetic field non vanishing even though one of them can be small. The case $\cos\alpha = 0$ can never be satisfied since we always assume non-vanishing integral charge n for D3-branes.

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have $\sin\theta \to 1$. We therefore requires $\cos\alpha \to 1$ for NCOS in the S-dual description. So, cases (1) and (2) will certainly not give NCOS in the S-dual theory. case (3) corresponds to the situation where the electric field times the closed string coupling is much smaller than the magnetic field in the original theory (version A), i.e. $|b_1| g_s \ll |b_2|$. So it is clear now that except when one of the field is much weaker than the other (or for purely electric or magnetic case) the NCYM is not related to NCOS by S-duality. Let us find out for each case above what is the S-dual theory of NCYM.

Case (1). We exclude the special case $\cos\alpha = 1$ which has been studied in [17]. We have the following scalings in the dual theory,

$$
\begin{aligned}
&\hat{G}_1 = \hat{G}_2 \sim \frac{1}{\alpha'}\,, \qquad \hat{\Theta}_1 \sim \alpha'^2\,, \qquad \hat{\Theta}_2 \sim \alpha'\,, \\
&\hat{G}_s \sim \frac{1}{\alpha'^2}\,, \qquad \hat{b}_1 \sim \frac{1}{\alpha'^2}\,, \qquad \hat{b}_2 \sim \frac{1}{\alpha'}\,.
\end{aligned} \tag{4.12}
$$

From the above we see that we have a field theory (since $\alpha'\hat{G}_1^{-1} = \alpha'\hat{G}_2^{-1} \sim \alpha'^2 \to 0$) according to (2.12) but defined in a commuting geometry.[12] But this theory is bad since it has an infinitely large open string coupling even though the singular metric can be brought to a non-singular one as we discussed in footnote 9. Both the electric field $\hat{b}_1 \sim 1/\alpha'^2$ (but $\tilde{\hat{E}} = \cos\alpha$ fixed) and magnetic field $\hat{b}_2 = 1/\alpha'$ (the $\tilde{\hat{B}} = 1/\alpha'$ still large) are infinitely large.

Case (2). In this case we have the following scalings:

$$
\hat{G}_1 = \hat{G}_2 \sim \alpha'^{-1+\delta/2}\,, \qquad \hat{\Theta}_1 \sim \alpha'^{2-\delta}\,, \qquad \hat{\Theta}_2 \sim \alpha'\,, \tag{4.13}
$$

with $\hat{b}_1, \hat{b}_2$ and $\hat{G}_s$ remain the same as in case (1). Recall that we have $\delta < 0$, so this case is not much different from case (1) as expected.

Case (3). This is the case where we expect to have NCOS. Let us list the relevant parameters,

$$
\alpha' \to 0\,, \qquad \hat{b}_1 = \frac{\tilde{b}}{2\pi G_s \alpha'^2}\left[1 - \frac{1}{2}\left(\frac{\alpha'}{\tilde{b}}\right)^2\right], \qquad \hat{b}_2 = \frac{1}{2\pi G_s \tilde{b}'}\left(\frac{\alpha'}{\tilde{b}'}\right)^{\beta-1},
$$

$$
\tilde{\hat{E}} = 1 - \frac{1}{2}\left(\frac{\alpha'}{\tilde{b}}\right)^2\left[1 + \left(\frac{\tilde{b}}{\tilde{b}'}\right)^2\left(\frac{\alpha'}{\tilde{b}'}\right)^{2(\beta-1)}\right],
$$

$$
\hat{G}_1 = \hat{G}_2 = \frac{\alpha'}{G_s \tilde{b}}\left[1 + \left(\frac{\tilde{b}}{\tilde{b}'}\right)^2\left(\frac{\alpha'}{\tilde{b}'}\right)^{2(\beta-1)}\right],
$$

[12] Since we assume $\tilde{x}^\mu$ to be fixed, $< \tilde{x}\tilde{x} > \sim 0$ just as in a ordinary Yang-Mills case in the decoupling limit. The coupling still blows up. We do not know how to make sense of such a theory.

$$
\hat{\Theta}_1 = \frac{2\pi G_s \tilde{b}}{\left[1 + (\tilde{b}/\tilde{b}')^2 (\alpha'/\tilde{b}')^{2(\beta-1)}\right]}, \qquad \hat{\Theta}_2 = \frac{2\pi G_s \tilde{b}}{\left[1 + (\tilde{b}/\tilde{b}')^2 (\alpha'/\tilde{b}')^{2(\beta-1)}\right]} \frac{\tilde{b}}{\tilde{b}'} \left(\frac{\alpha'}{\tilde{b}'}\right)^{\beta-1},
$$

$$
\hat{G}_s = \frac{\left[1 + (\tilde{b}/\tilde{b}')^2 (\alpha'/\tilde{b}')^{2(\beta-1)}\right]}{G_s}. \tag{4.14}
$$

From above, we have basically three sub cases depending on the range of the parameter β. Remember that we always have $\tilde{\tilde{E}} \to 1$ as $\alpha' \to 0$. For $\beta \geq 1$, we have $\alpha' \hat{G}_1^{-1} = \alpha' \hat{G}_2^{-1} =$ fixed, and the open string coupling $\hat{G}_s$ and at least one of the non-commutative parameters are also fixed. Therefore we end up with a NCOS. However, the $\beta > 1$ case differs from $\beta = 1$ case in that the former has only non-commutativity along space-time directions and the open string coupling is inversely related to its S-dual, while the latter has noncommutativity not only in space-time directions but also in space-space directions and the relation between the open string couplings related by S-duality has been modified. We would like to stress that for $\beta > 1$, the parameters such as the metric, non-commutative parameters and the open string coupling which define the NCOS theory are independent of the parameter $\beta > 1$. The same is true for the gravity dual description which will be discussed in the following section. This indicates that many of the scaling limits corresponding to different $\beta > 1$ are actually equivalent. This $b > 1$ case corresponds to the $\delta' > 0$ case discussed in the previous subsection. The precise relation between this β and the δ and δ' discussed in the previous subsection will be given in section 6. For $\beta = 1$, corresponding to $\delta' = 0$ discussed in the previous subsection, from the above equation (4.14), once again we have $\hat{G}_s G_s = 1 + (\tilde{b}/\tilde{b}')^2 > 1$. This implies that in the present case a weakly coupled theory cannot be obtained from a strongly coupled theory by S-duality unless the strongly coupled theory has a coupling greater than $1 + (\tilde{b}/\tilde{b}')^2$. For $0 < \beta < 1$, we end up with an ill-behaved field theory as in cases (1) and (2) discussed above with the open string coupling blowing up even though we now have reached the critical electric field limit.[13] Unlike in cases (1) and (2), the singular metric here cannot be brought to a non-singular one following the description given in footnote 9.

[13]We would like to stress that the present discussion, i.e. NCYM and its S-duality or case II, is independent of what we have discussed in case I. In particular, many parameters used here are different, in definition, from those used in case I even though we often use the same symbols. For example, the parameters $\tilde{b}$ and $\tilde{b}'$ here are different from those used in case I. Even the string constant α' may be different. But we expect that the conclusions drawn under the same conditions should be the same, for example, we always have $\hat{G}_s G_s > 1$ in the presence of both electric and magnetic fields. Another example is that in case I, the limit $\tilde{b}' \to \tilde{b}$ gives pure electric case while here the corresponding limit is $\tilde{b}/\tilde{b}' \to$ zero where we should take $\tilde{b}' \to \infty$ while keeping $\tilde{b}$ fixed. These two cases can be identified if we exchange $\cos\theta \leftrightarrow \cos\alpha$ which can be recognized through the respective decoupling limits and the S-duality. Some relations between the parameters in the two cases will be further clarified in section 6.

The decoupling of NCOS for $\beta \geq 1$ from the closed strings can be discussed similarly as we did in the previous subsection and we will not repeat it here.

The above discussion indicates that the effect of one of the fields on the other really drops out if $\beta > 1$. This effect becomes important for $\beta = 1$, a sort of critical value. As $\beta < 1$, the open string theory flows to an ill-defined strongly-coupled field theory. Thus β plays the role of some kind of an order parameter.

With this understanding of the relationship between NCYM and NCOS, we will present the gravity dual descriptions for each case discussed here in the following section.

5. The gravity dual descriptions

Here we present the gravity dual of the NCOS discussed first (case I) in the previous section. The decoupling limit can be collectively given as,

$$\begin{aligned}
&\alpha' \longrightarrow 0\,, \qquad u = \frac{r}{\sqrt{\tilde{b}'}\sqrt{\alpha'}} = \text{fixed}\,, \qquad g_1 = \left(\frac{\alpha'}{\tilde{b}'}\right)^{\delta}, \qquad g_2 = \frac{\alpha'}{\tilde{b}}\,, \\
\cos\alpha &= \left(\frac{\alpha'}{\tilde{b}'}\right)^{(1-\delta)/2}, \qquad \cos\theta = \frac{\tilde{b}}{\tilde{b}'}\,, \\
R^4 &= \text{fixed} = 4\pi g_s n \frac{\cos\alpha}{\cos\theta} = 4\pi G_s n\,,
\end{aligned} \tag{5.1}$$

where the parameter $\delta < 1$. The closed string coupling can be obtained from the last expression in (5.1) as,

$$g_s = \frac{G_s \tilde{b}}{\tilde{b}'}\left(\frac{\tilde{b}'}{\alpha'}\right)^{\frac{1-\delta}{2}}. \tag{5.2}$$

Under this decoupling limit the harmonic functions in (3.9) take the following form:

$$H = \frac{R^4}{u^4 \tilde{b}'^2}\left(\frac{\tilde{b}'}{\alpha'}\right)^{1-\delta}, \qquad H' = 1 + \frac{R^4 \tilde{b}^2}{u^4 \tilde{b}'^4}\,, \qquad H'' = 1 + \frac{R^4}{u^4 \tilde{b}'^2}\,. \tag{5.3}$$

So the metric, dilaton, axion, B-field and the RR 2-form now reduce to

$$\begin{aligned}
ds^2 &= \alpha' \bar{h}^{-1/2}\left\{\frac{u^2}{R^2}\left[-(d\tilde{x}^0)^2 + (d\tilde{x}^1)^2 + \bar{h}'\left((d\tilde{x}^2)^2 + (d\tilde{x}^3)^2\right)\right] + R^2\left[\frac{du^2}{u^2} + d\Omega_5^2\right]\right\}, \\
e^{2\phi} &= G_s^2 \frac{\tilde{b}^2}{\tilde{b}'^2}\frac{\bar{h}'}{\bar{h}^2}\,, \qquad \chi = \frac{\sqrt{\tilde{b}'^2 - \tilde{b}^2}}{\tilde{b} G_s}\bar{h}\,, \\
2\pi\alpha' B &= \alpha' \frac{u^4 \tilde{b}'}{R^4} d\tilde{x}^0 \wedge d\tilde{x}^1 - \alpha' \frac{\sqrt{\tilde{b}'^2 - \tilde{b}^2}}{\tilde{b}}\frac{u^4 \tilde{b}'}{R^4}\bar{h}' d\tilde{x}^2 \wedge d\tilde{x}^3\,, \\
A_2 &= \alpha'^2 \frac{u^4}{g_1 G_s R^4}\frac{\sqrt{\tilde{b}'^2 - \tilde{b}^2}}{\tilde{b}} d\tilde{x}^0 \wedge d\tilde{x}^1 + \alpha'\left(\frac{\tilde{b}'}{\tilde{b}}\right)^3 \frac{\tilde{b} u^4}{R^4 G_s}\bar{h}' d\tilde{x}^2 \wedge d\tilde{x}^3\,,
\end{aligned} \tag{5.4}$$

where we have defined $\bar{h} = \frac{1}{1+a^4u^4}$, $\bar{h}' = \frac{1}{\frac{\tilde{b}^2}{\tilde{b}'^2}+a^4u^4}$ with $a^4 = \frac{\tilde{b}'^2}{R^4}$. As one can see that g_1 appears only in the 01-component of the RR two-form A_2, indicating that the scaling behaviour of this component is dependent on the parameter δ. However, this component measured in terms of α' is proportional to $\alpha'^{1-\delta}$ which vanishes as $\alpha' \to 0$ since $1-\delta > 0$. So the gravity dual description is independent of the $\delta < 1$.

Note from the form of the metric in (5.4) that as $u \to 0$, the metric reduces to $AdS_5 \times S^5$ form as expected and it starts deviating from this form at $u \sim \frac{R\sqrt{\tilde{b}}}{\tilde{b}'}$. We also notice from (5.4) that as $u \to 0$, $e^{2\phi} \to G_s^2$ and so the open string coupling is the same as the closed string coupling at the IR. This occurs always as noticed in [28, 15]. Also the axion χ at the IR does not vanish but reaches a constant value $(\tilde{b}'^2 - \tilde{b}^2)^{1/2}/(G_s\tilde{b})$. The S-dual dilaton is related to the original one through the relation $e^{\hat{\phi}} = e^{\phi}(\chi^2 + e^{-2\phi})$. At the IR, we have $e^{\hat{\phi}} = \tilde{b}'^2/(G_s\tilde{b}^2)$. But the closed string coupling $e^{\hat{\phi}}$ at the IR gives the open string coupling $\hat{G}_s$. So we provide an explanation for the S-dual coupling relation, derived in the previous section, using the S-duality of type-IIB string theory. This is another way to understand why in the presence of both electric and magnetic fields, a strongly coupled theory does not necessarily give a weakly coupled S-dual theory. It is easy to check that if $\tilde{b} = \tilde{b}'$, i.e. one of the field vanishes, then $\chi = 0$ at the IR and we recover the simple inverse relation for the couplings. Note, however, that at UV the closed string coupling blows up.

Now we move on to give the S-dual of the above gravity dual description of NCOS, i.e. the gravity dual of NCYM. Using (3.11), (3.12), (5.1) and (5.3), we have

$$\begin{aligned} d\hat{s}^2 &= \alpha' \frac{\tilde{b}'}{G_s\tilde{b}} \left\{ \frac{u^2}{R^2}\left[-(d\tilde{x}^0)^2 + (d\tilde{x}^1)^2 + \bar{h}'\left((d\tilde{x}^2)^2 + (d\tilde{x}^3)^2\right)\right] + R^2\left[\frac{du^2}{u^2} + d\Omega_5^2\right]\right\}, \\ e^{\hat{\phi}} &= \frac{\tilde{b}'}{G_s\tilde{b}}\bar{h}'^{1/2}, \qquad \hat{\chi} = -\frac{G_s\tilde{b}}{\tilde{b}'}\frac{\sqrt{\tilde{b}'^2 - \tilde{b}^2}}{\tilde{b}'}, \end{aligned} \tag{5.5}$$

where $\bar{h}'$ is the same as that defined for NCOS. We also have $2\pi\alpha'\hat{B} = A_2$ and $\hat{A}_2 = -2\pi\alpha' B_2$ with B_2 and A_2 given in (5.4). At the IR, i.e. for $u \to 0$, the metric describes $AdS_5 \times S^5$ as expected. The closed string coupling is now the same as the open string coupling $\hat{G}_s$. The axion $\hat{\chi}$ for the present case is independent of the u. In this case $e^{\hat{\phi}} \to 0$ in UV limit.

In the previous section, we concluded for case I that the S-dual of NCOS always corresponds to NCYM, from the open string viewpoint. In the above, we have shown that this is also true from the gravity (or closed string) viewpoint of D-branes. In terms of the effective description of the closed strings discussed in footnote 9, the above metric keeps the same form except for replacing the α' by the effective $\alpha'_{\rm eff}$. This can be checked easily as follows. Effectively, the new closed string metric is related to the old one given in (5.5) as $d\hat{s}^2_{\rm eff} = d\hat{s}^2/(\alpha'/\tilde{b}')^{(1+\delta)/2}$. In other words, the α' in metric $d\hat{s}^2$ is replaced by $\tilde{b}'(\alpha'/\tilde{b}')^{(1-\delta)/2}$ which is just the $\alpha'_{\rm eff}$ defined in footnote 9. In the previous section, we also showed that the story for NCYM is different. The S-dual of NCYM is not necessarily an NCOS theory. We will show that the same conclusion can be drawn from the gravity consideration as well in the following.

Let us present first the gravity dual description of the NCYM for case II discussed in the previous section. We list the decoupling limit collectively as

$$\begin{aligned} \alpha' \to 0\,, \qquad & u = \frac{r}{\alpha'} = \text{fixed}\,, \qquad g_2 = \left(\frac{\alpha'}{\tilde{b}}\right)^2, \qquad \cos\theta = \frac{\alpha'}{\tilde{b}}\,, \\ g_1 \cos^2\alpha \;=\; 1\,, \qquad & R^4 = \text{fixed} = 4\pi g_s n \frac{\cos\alpha}{\cos\theta} = 4\pi G_s n\,. \end{aligned} \tag{5.6}$$

As is understood, the parameters $\tilde{b}$ and $\tilde{b}'$ (introduced later in the discussion of the S-duality of the NCYM) here are different, in definition, from those in the previous case. The closed string coupling g_s is related to the fixed open string coupling G_s as given in (5.6). The harmonic functions take the form;

$$H = \frac{g_1 R^4}{u^4 \alpha'^2}\,, \qquad H' = 1 + \frac{R^4}{\tilde{b}^2 u^4}\,, \qquad H'' = \frac{R^4}{\alpha'^2 u^4}\,. \tag{5.7}$$

So the metric, dilaton, axion, 2-form B-field and the RR 2-form A_2 in eqs. (3.8) reduce to,

$$\begin{aligned} ds^2 &= \alpha' \left\{ \frac{u^2}{R^2} \left[-(d\tilde{x}^0)^2 + (d\tilde{x}^1)^2 + \bar{h}\left((d\tilde{x}^2)^2 + (d\tilde{x}^3)^2\right) \right] + R^2 \left[\frac{du^2}{u^2} + d\Omega_5^2 \right] \right\}, \\ e^{2\phi} &= G_s^2 \bar{h}\,, \qquad \chi = \frac{\tilde{b}}{\alpha'} G_s^{-1} \sin\alpha\,, \\ 2\pi\alpha' B &= \alpha'^2 \frac{u^4 \sin\alpha}{R^4} d\tilde{x}^0 \wedge d\tilde{x}^1 - \alpha' \frac{\tilde{b} u^4}{R^4} \bar{h}\, d\tilde{x}^2 \wedge d\tilde{x}^3\,, \\ A_2 &= \alpha' \frac{\tilde{b} u^4}{g_1 G_s R^4} d\tilde{x}^0 \wedge d\tilde{x}^1 + \frac{\tilde{b}^2 u^4 \sin\alpha}{G_s R^4} \bar{h}\, d\tilde{x}^2 \wedge d\tilde{x}^3\,, \end{aligned} \tag{5.8}$$

where we have defined $\bar{h} = \frac{1}{1 + a^4 u^4}$ with $a^4 = \frac{\tilde{b}^2}{R^4}$ and g_1 is related to $\sin\alpha$ as $g_1 = 1/(1 - \sin^2\alpha)$.

Again we note that the metric in (5.8) has $AdS_5 \times S^5$ form as $u \to 0$ as expected and its form starts deviating from $AdS_5 \times S^5$ at $u \sim R/\sqrt{\tilde{b}}$. From (5.8) we find that as $u \to 0$, $e^{2\phi} = G_s^2$ and so, the open string coupling is again the same as the closed string coupling at the IR. As $u \to \infty$, $e^{2\phi} \to 0$. Notice that the metric and dilaton are independent of the scaling factor g_1 but all the other fields do depend on g_1. As we can see from the above, the 01-component of B-field depends, i.e. the electric field depends on g_1. But this field plays no role for the decoupling limit since its ratio to the magentic field is like $\sim \alpha'$, vanishing in the decoupling limit. Notice that the axion is constant, independent of u. However, it blows up in the decoupling limit unless $\sin\alpha/\alpha' \to 0$ or fixed value. The behavior of axion is very important in determining whether we have good or bad S-dual theory. This is because under S-duality, the closed string coupling in the dual theory is determined by the dilaton and the axion in the original theory as $e^{\hat{\phi}} = e^{\phi}(\chi^2 + e^{-2\phi})$. So if χ blows up with a fixed e^{ϕ} this will imply that the S-dual closed string coupling $e^{\hat{\phi}}$ always blows up in the decoupling limit. This in turn implies that the open string coupling blows up since it is the same as the closed string coupling at the IR. If this happens, we have neither a good field theory nor a good gravity dual description. We have shown the former in the previous section. We will show the case for the gravity description in the following. So the behavior of the axion in the original theory determines precisely whether we have a good S-dual theory or not even though the original theory is nicely behaved (which is NCYM here). The behavior of the RR 2-form potential is also consistent with this even though it is not directly relevant to the perturbative open string dynamics, i.e. NCYM, in the decoupling limit.

Keeping this in mind we will now discuss the S-duality of the gravity dual description of the NCYM discussed above. As in the previous section (case II), we have three cases here also depending on the angle α or the scaling factor g_1:

(1) $g_1 = \text{fixed} \neq 1$,

(2) $\cos\alpha \to 0$ as $\alpha' \to 0$,

(3) $\sin\alpha \to 0$ as $\alpha' \to 0$.

Case (3) can also be divided into three subcases which are determined by the order parameter β defined as $\sin\alpha = (\alpha'/\tilde{b}')^{\beta}$. As studied in the previous section, cases (1), (2) and $0 < \beta < 1$ in case (3) all give ill behaved but ordinary YM theories. The gravity description for these cases can be given in a unified way as

$$d\hat{s}^2 = \frac{\tilde{b}\sin\alpha}{G_s}\left\{\frac{u^2}{R^2}\left[-(d\tilde{x}^0)^2 + (d\tilde{x}^1)^2 + \bar{h}\left((d\tilde{x}^2)^2 + (d\tilde{x}^3)^2\right)\right] + R^2\left[\frac{du^2}{u^2} + d\Omega_5^2\right]\right\},$$
$$e^{\hat{\phi}} = \frac{\sin^2\alpha}{G_s}\left(\frac{\tilde{b}}{\alpha'}\right)^2 \bar{h}^{1/2}, \qquad \hat{\chi} = -\frac{G_s}{\sin\alpha}\frac{\alpha'}{\tilde{b}}, \tag{5.9}$$

where all the quantities are defined as in (3.11) and again $2\pi\alpha'\hat{B} = A_2$ and $\hat{A}_2 = -2\pi\alpha' B$ with B and A_2 given in (5.8). For each of these three cases, we can check that the closed string coupling always blows up as $\alpha' \to 0$ precisely because $\sin\alpha/\alpha' \to \infty$ as $\alpha' \to 0$ as discussed above. So the gravity description breaks down. At the IR, the closed string coupling $e^{\hat{\phi}}$ is just the open string coupling given in the previous section which also blows up. So the underlying theory is not good even though the original NCYM theory appears fine.

For other two sub-cases in case (3), i.e. for $\beta \geq 1$, we have good gravity descriptions as expected. They can also be given in a unified way as

$$d\hat{s}^2 = \alpha' \left(\frac{\hat{G}_s}{G_s}\right)^{1/2} \bar{h}'^{-1/2} \left\{ \frac{u^2}{R^2} \left[-d\tilde{x}_0^2 + d\tilde{x}_1^2 + \bar{h}\left(d\tilde{x}_2^2 + d\tilde{x}_3^2\right)\right] + R^2 \left[\frac{du^2}{u^2} + d\Omega_5^2\right]\right\},$$
$$e^{\hat{\phi}} = \hat{G}_s \frac{\bar{h}^{1/2}}{\bar{h}'}, \qquad \hat{\chi} = -\hat{G}_s^{-1} \frac{\tilde{b}}{\tilde{b}'} \left(\frac{\alpha'}{\tilde{b}'}\right)^{\beta-1} \bar{h}', \tag{5.10}$$

where we have defined $\bar{h}' = 1/(1 + a'^4 u^4)$ with $a'^4 = \tilde{b}^2/(R^4 \hat{G}_s G_s)$ and $\bar{h}$ is the same as that given for NCYM. We have $\hat{G}_s G_s = [1 + (\tilde{b}/\tilde{b}')^2]$, for $\beta = 1$ and $\hat{G}_s G_s = 1$, for $\beta > 1$. Again the metric reduces to $AdS_5 \times S^5$ as expected and its form starts deviating from $AdS_5 \times S^5$ at $u \sim R/\sqrt{\tilde{b}}$. The open string coupling $\hat{G}_s$ is the same as the closed string coupling at the IR. Unlike in the original gravity description, the closed string coupling blows up at the UV.

6. A few remarks

In the previous sections, we have studied the relationship between NCOS and NCYM using the BPS ((F, D1), D3) bound state configuration in two versions related by S-duality. For concreteness, we choose the asymptotic string-frame metric in version A as $\eta_{MN} = (-, +, \ldots, +)$, where $M, N = 0, 1, \ldots, 9$ with respect to the unscaled coordinates. Because of this choice, the string-frame metric in the S-dual version (i.e. version B) is not η_{MN} asymptotically but to $g_s^{-1}\eta_{MN}$ with respect to the unscaled coordinates.

In the previous section, we have given the gravity dual descriptions of NCOS (or NCYM) and its S-dual in case I (or case II). In case I, we always have $r \sim \sqrt{\alpha'}u$ while in case II we have $r = \alpha' u$. The scalings for r are different in the two cases.[14] How can we understand this difference? Another puzzle is: when we discuss the condition in version B for NCOS in case II as the S-dual of NCYM in version A, we have (see eq. (4.14))

$$\frac{\hat{b}_1}{\hat{b}_2} \sim \frac{1}{\alpha'^{1+\beta}}, \tag{6.1}$$

with $\beta \geq 1$. While in case I for NCOS in version A, we have

$$\frac{b_1}{b_2} \sim \frac{g_1 \sin\alpha}{g_2 \tan\theta} \sim \frac{1}{\alpha'^{1-\delta+\delta'}}, \tag{6.2}$$

where the last equality is obtained using eqs. (4.2)–(4.4) and we also use $\sin\theta \sim \alpha'^{\delta'}$ with $\delta' \geq 0$. These two criteria should be the same but from their appearances, it does not seem to be the case. Let us understand these two puzzles at the same time.

[14] We do not have this difference if we study the NCYM in version B (rather than in version A) in case II. However, the above choice helps us to understand things better and that is why we prefer to present in this way.

For convenience, let us collect some scalings here: for case I, we have

$$r \sim \sqrt{\alpha'} u\,, \qquad g_1 \sim \alpha'^{\delta}\,, \qquad g_2 \sim \alpha'\,, \qquad \sin\theta \sim \alpha'^{\delta'}\,, \qquad g_s^{-1} \sim \alpha'^{(1-\delta)/2}\,, \quad (6.3)$$

where $\delta < 1$ and $\delta' \geq 0$. While for case II, we have

$$r = \alpha' u\,, \qquad g_1 \sim 1\,, \qquad g_2 \sim \alpha'^2\,, \qquad g_s \sim \alpha'\,, \qquad \sin\alpha \sim \alpha'^{\beta}\,, \qquad (6.4)$$

where $\beta \geq 1$.

The string constant α' is a property of the string, independent of the background in which it moves. However, the scaling behavior for the radial coordinate r in a gravity dual description in terms of α' does depend on the asymptotic metric. We also know that if α' is the string constant defined through $1/(4\pi\alpha') \int \partial X^M \partial X^N \eta_{MN}$ and r is one of the coordinates X^M, then for NCYM, we should have $r = \alpha' u$ while for NCOS, we should have $r = \sqrt{\alpha'}\, u$ [28, 17]. So for the NCOS in case I (in version A) and for NCYM in case II (also in version A), we do have the correct scaling behavior. They can be further understood in the following way. For the NCOS, the string lies along x^1-direction and the α' is defined with respect to the unscaled x^1. With respect to the scaled $\tilde{x}^1$, we have $1/(4\pi\alpha') \int \partial x^1 \partial x^1 = 1/(4\pi\alpha') \int \partial\tilde{x}^1 \partial\tilde{x}^1 g_1$. So the effective string constant α' is $\alpha'_{\rm eff} = \alpha' g_1^{-1}$. In order to have the same effective $\alpha'_{\rm eff}$ for the radial coordinate r, we must rescale $r = \sqrt{g_1}\tilde{r}$. We expect that $\tilde{r} \sim \sqrt{\alpha'_{\rm eff}}\, u$. It is easy to check, using the above relations, that this is indeed true. For the NCYM, $g_1 \sim 1$ and we always have $r \sim \alpha' u$. So for either the NCOS or the NCYM, the scaling behavior remains the same whether we use α' or $\alpha'_{\rm eff}$. With the above in mind, we now try to understand $r \sim \sqrt{\alpha'}\, u$ for NCYM in case I and $r = \alpha' u$ for NCOS in case II, both of them in version B.

The asymptotic metric for g_{11} with respect to the scaled coordinate in version B is $g_s^{-1} g_1$. So we have $1/(4\pi\alpha') \int \partial x^1 \partial x^1 g_s^{-1} = g_s^{-1} g_1/(4\pi\alpha') \int \partial\tilde{x}^1 \partial\tilde{x}^1$. So the effective string constant $\alpha'_{\rm eff} = \alpha' g_s/g_1$. In order to have the same effective constant for coordinate r, we must again rescale the r as $\tilde{r} = r/\sqrt{g_1}$.

For the NCYM in case I, we apply the above two relations and we have $\alpha'_{\rm eff} \sim \alpha'^{(1-\delta)/2}$ where eq. (6.3) has been used, which is also consistent with that given in footnote 9. This gives the rescaled $\tilde{r} \sim \sqrt{\alpha'} u/\sqrt{g_1} \sim \alpha'^{(1-\delta)/2} u \sim \alpha'_{\rm eff} u$, the expected relation. While for the NCOS in case II, we have $\alpha'_{\rm eff} \sim \alpha'^2$. This gives $\tilde{r} = \alpha' u \sim \sqrt{\alpha'_{\rm eff}}\, u$, again the correct relation.

Even though we discuss the scaling of r for both case I and case II at the same time, so far we have not brought any connection between them yet. Even then we expect that the conclusions drawn in each case should be the same. But the ratio between the electric and magnetic fields, determining NCOS, looks different in the two cases. How can we resolve this puzzle? Note that α' used in defining the ratio in case I can be taken as the effective string constant with respect to the unscaled coordinates for the NCOS in this case while the α' used in defining the same ratio in case II can be taken as the effective string constant with respect to the unscaled coordinates for the NCYM in that case. In other words, the α' is taken as the effective string constant for different theories in the two cases. This is the source of difference. We expect that if the same α' is used as an effective string coupling for either NCYM or NCOS in both case I and case II, the ratio should scale the same way. This suggests that if expressed in terms of the effective string constant $\alpha'_{\rm eff} \sim \alpha'^{(1-\delta)/2}$ for the NCYM in case I, the ratio should scale in the same way as that in case II. Let us check if this is true. With $\alpha'_{\rm eff} \sim \alpha'^{(1-\delta)/2}$, we have the ratio in case I, from (6.2), as

$$\frac{b_1}{b_2} \sim \frac{1}{\alpha'^{\,2+2\delta'/(1-\delta)}_{\rm eff}}\,, \tag{6.5}$$

with $\delta < 1, \delta' \geq 0$. Since $\delta < 1$, so $\alpha'_{\rm eff} \to 0$ as $\alpha' \to 0$. This ratio has the same behavior as that in case II if we identify the α' there as our present effective $\alpha'_{\rm eff}$ and β there as $\beta = 1 + 2\delta'/(1-\delta)$. Note that $\beta \geq 1$ is consistent with $1+2\delta'/(1-\delta) \geq 1$, since $\delta' \geq 0$ and $\delta < 1$. In particular, $\beta = 1$ corresponds to $\delta' = 0$, both of which give their respective NCOS's with noncommutativity in both space-space and space-time directions.

To identify the two cases completely, we need to set $g_1 \sim 1$ for NCYM in version A of case II the same as $g_s g_1 \sim \alpha'^{(1+\delta)/2}$ for NCYM in version B of case I. This implies that $\delta = -1$. Now $\alpha'_{\rm eff} = \alpha'$ as expected. This is consistent with the fact that the NCYM in version A of case II has fixed metric while the NCYM in version B of case I has fixed metric only for $\delta = -1$. One can check that the scalings of the relevant quantities for NCOS in version A of case I are the same as those for the NCOS in version B of case II (similarly for the NCYM's in both cases). For example, $g_2 \sim \alpha'$ for the NCOS in version A of case I while the equivalent one for the NCOS in version B of case II is $g_s^{-1} g_2 \sim \alpha'^{-1}\alpha'^2 \sim \alpha'$ as expected.

Finally in this section, we discuss the possible quantization for the open string coupling constant G_s and $\hat{G}_s$ in the decoupling limit. One can use the definitions for $\cos\theta$ and $\cos\alpha$ in (3.10) to express the open string coupling G_s as

$$G_s = g_s \frac{\cos\alpha}{\cos\theta} = g_s \frac{q^2 + n^2}{n\sqrt{(pg_s)^2 + q^2 + n^2}}\,, \tag{6.6}$$

and the S-dual $\hat{G}_s$ in (4.8) as

$$\begin{aligned} \hat{G}_s &= g_s^{-1}\left(1 - \sin^2\theta\cos^2\alpha\right)^{1/2}\left(1 + \frac{\tan^2\alpha}{\cos^2\theta}\right)^{1/2} \\ &= g_s^{-1}\frac{\sqrt{(pg_s)^2 + n^2}}{\sqrt{(pg_s)^2 + q^2 + n^2}}\left(1 + \frac{(pg_s)^2}{n^2}\right)^{1/2}. \end{aligned} \tag{6.7}$$

The integral charge n for D3-branes is always fixed. In the case $pg_s \gg \sqrt{q^2+n^2}$, we have

$$G_s = \frac{q^2+n^2}{np}, \qquad \hat{G}_s = \frac{p}{n}. \tag{6.8}$$

In other words, both G_s and $\hat{G}_s$ are independent of the closed string coupling and quantized in the above limit. The question is now: can both G_s and $\hat{G}_s$ be finite and be related to the decoupling limits discussed in the previous sections?

Let us discuss the case I first. To the NCOS, the integer p is related to the electric field and the integer q is related to the magnetic field while to its S-dual, i.e. NCYM, these relations are exchanged, i.e. q to the electric field and p to the magnetic field. The decoupling limit for this case says that $\cos\alpha \to 0, \cos\theta \sim 1$ and $g_s \to \infty$. They imply that $q \sim n =$ fixed and $pg_s \gg \sqrt{n^2+q^2}$, the right condition to validate the quantizations of both G_s and $\hat{G}_s$ while at the same time, p can be finite. So the decoupling limit in case I for NCOS/NCYM gives finite and quantized open string coupling G_s and its S-dual $\hat{G}_s$. Also if $q = 0$, i.e. one of the field vanishes, we have $\hat{G}_s G_s = 1$, the expected relation.

When NCYM and NCOS are S-dual to each other in case II, we have $\cos\theta \to 0, \cos\alpha \sim 1$ and $g_s \to 0$. This decoupling limit does not give the condition needed for G_s and $\hat{G}_s$ to be independent of the closed string coupling. Therefore both G_s and $\hat{G}_s$ cannot be expressed purely in terms of the charges p, q and n. However, for the subcase 2 in case II for which the S-dual of the NCYM is a singular ordinary Yang-Mills, we have $\cos\theta \to 0, \cos\alpha \to 0$ which implies that $pg_s \gg \sqrt{q^2+n^2}$ and $q \gg n$. In this case, we also have both G_s and $\hat{G}_s$ to be independent of the closed string coupling g_s and they are given as

$$G_s = \frac{q^2}{pn}, \qquad \hat{G}_s = \frac{p}{n}, \tag{6.9}$$

which gives $G_s\hat{G}_s = q^2/n^2 \gg 1$, as expected.

Let us now try to understand when NCOS and NCYM are S-dual to each other, why in case I the open string couplings are independent of the closed string coupling while in case II they are not? Our possible explanation is as follows: in case I, we choose the fundamental string-frame in version A and we end up with NCOS as a fundamental open string because of the critical electric field. In other words, the tension for the D3-branes in the decoupling limit for NCOS must be independent of the closed string coupling. Therefore, the corresponding gauge coupling is also independent of the closed string coupling, which in turn implies that the open string coupling is independent of the closed string coupling. This can also be understood for its S-dual, i.e. NCYM, in version B of case I. Given that the version A is expressed in the fundamental string frame, version B is therefore expressed actually in D-string frame. So for a D-string in version B, its tension is independent of the closed string coupling.[15] Since the S-dual of NCOS is NCYM with a rank 2 B-field, this NCYM is equivalent to an ordinary Yang-Mills in $1+1$ dimensions with U(∞) gauge group whose gauge coupling is determined by the D-string tension [15]. But this D-string tension in version B is independent of the closed string coupling. So the open string coupling must be also independent of the closed string coupling.

With the above, it is not difficult to understand why in case II, the open string couplings are dependent on the closed string coupling since now the NCOS as a fundamental string moves in a D-string frame while NCYM is defined in a fundamental string frame.

Note added. After the submission of this paper, there were two related papers which appeared in the net [32, 33].

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